AN INTRODUCTION TO THE VIBRATION RESPONSE SPECTRUM
Revision D

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INTRODUCTION

Mechanical shock pulses are often analyzed in terms of shock response spectra, as discussed in Reference 1. The shock response spectrum assumes that the shock pulse is applied as a common base input to an array of independent single-degree-of-freedom systems. The shock response spectrum gives the peak response of each system with respect to the natural frequency. Damping is typically fixed at a constant value, such as 5%, which is equivalent to an amplification factor of Q=10.

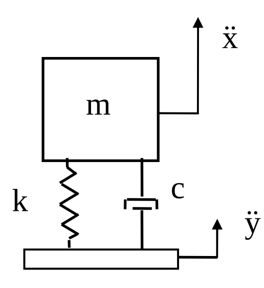
A similar tool is available for vibration. This tool is the *vibration response spectrum.* This function gives the root-mean-square response of each system to an acceleration base input.[1] The base input is an acceleration power spectral density.

The vibration response spectrum is particularly suited for random vibration inputs. Pure sinusoidal vibration, on the other hand, can be dealt with using time domain methods.

The vibration response spectrum has many uses. The purpose of this tutorial is to present this function and give an example of a typical application.

EQUATION OF MOTION

Consider a single degree-of-freedom system



where m equals mass, c equals the viscous damping coefficient, and k equals the stiffness. The absolute displacement of the mass equals x, and the base input displacement equals y.

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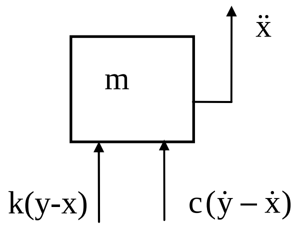
[1] A vibration response spectrum may also be given in terms of the 3σ amplitude. The 3σ value is equal to three times the root-mean-square value assuming that the mean value is zero.

The free-body diagram is



Summation of forces in the vertical direction,

$$\sum F = m\ddot{x} \tag{1a}$$

$$m\ddot{x} = c(\dot{y}-\dot{x}) + k(y-x) \tag{1b}$$

Define a relative displacement

$$z = x - y \tag{2}$$

Substituting the relative displacement terms into equation (1b) yields

$$m(\ddot{z}+\ddot{y}) = -c\dot{z} - kz \tag{3}$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4}$$

Dividing through by mass yields,

$$\ddot{z} + (c/m)\dot{z} + (k/m)z = -\ddot{y} \tag{5}$$

By convention,

$$(c/m) = 2\xi\omega_n$$
$$(k/m) = \omega_n{}^2$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substituting the convention terms into equation (5) yields

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z = -\ddot{y} \tag{6}$$

The derivation is continued in Appendix A. The derivation is based on References 2 and 3.

SOLUTION APPROACHES

Options

There are two approaches to solving equation (6) for the case where the excitation is in the form of a base input power spectral density. Each approach yields an equation for the root-mean-square acceleration response of the mass.

Miles Equation

The first approach is called Miles equation. This approach yields a simple equation for the absolute acceleration response $\ddot{x}_{GRMS}$.

$$\ddot{x}_{GRMS}(f_n,\xi) = \sqrt{\left(\frac{\pi}{2}\right)\left(\frac{f_n}{2\xi}\right)\hat{Y}_{APSD}(f_n)} \tag{7}$$

where

f_n is the natural frequency,

$\hat{Y}_{APSD}(f_n)$ is the base input acceleration power spectral density.

Note that the base input amplitude is taken at the natural frequency. The derivation assumes that this amplitude is constant across the entire frequency domain, from zero to infinity.

Miles equation should only be used if the power spectral density amplitude is flat within one octave on either side of the natural frequency.

General Approach

The second approach allows the power spectral density to vary with frequency and to be defined over a finite frequency domain.

The resulting response equation is

$$\ddot{x}_{GRMS}(f_n,\xi) = \sqrt{\sum_{i=1}^{N} \frac{\left\{1+(2\xi\rho_i)^2\right\}}{\left\{\left[1-\rho_i^2\right]^2+\left[2\xi\rho_i\right]^2\right\}}\hat{Y}_{APSD}(f_i)\Delta f_i}\, , \rho_i = f_i / f_n \tag{8}$$

Equation (8) appears cumbersome. In reality, it can be easily implemented via a computer program. The remainder of this report will use this general equation rather than the restrictive Miles equation.

STATISTICS

The vibration response spectrum is typically represented in terms of Acceleration Response (GRMS) versus Natural Frequency (Hz).

Note that the GRMS value is equal to the standard deviation assuming zero mean. The standard deviation is represented as 1σ.

The response time history has the probability characteristics shown in Table 1 assuming a normal distribution.

Table 1.
Statistical Probabilities for a Normal Distribution

Probability inside	±1σ Limits = 68.27%
Probability outside	±1σ Limits = 31.73%
Probability inside	±3σ Limits = 99.73%
Probability outside	±3σ Limits = 0.27%

MIL-STD-1540C EXAMPLE

Avionics components must be subjected to random vibration tests to verify the integrity of parts and workmanship. The components are mounted to a shaker table for this testing. The components are typically powered and monitored during these tests.

The test specifications may come from established standards or from measured flight data. An example of a power spectral density specification from MIL-STD-1540C is shown in Figure 1 and in Table 2. This level is applied as a base input to the avionics component.

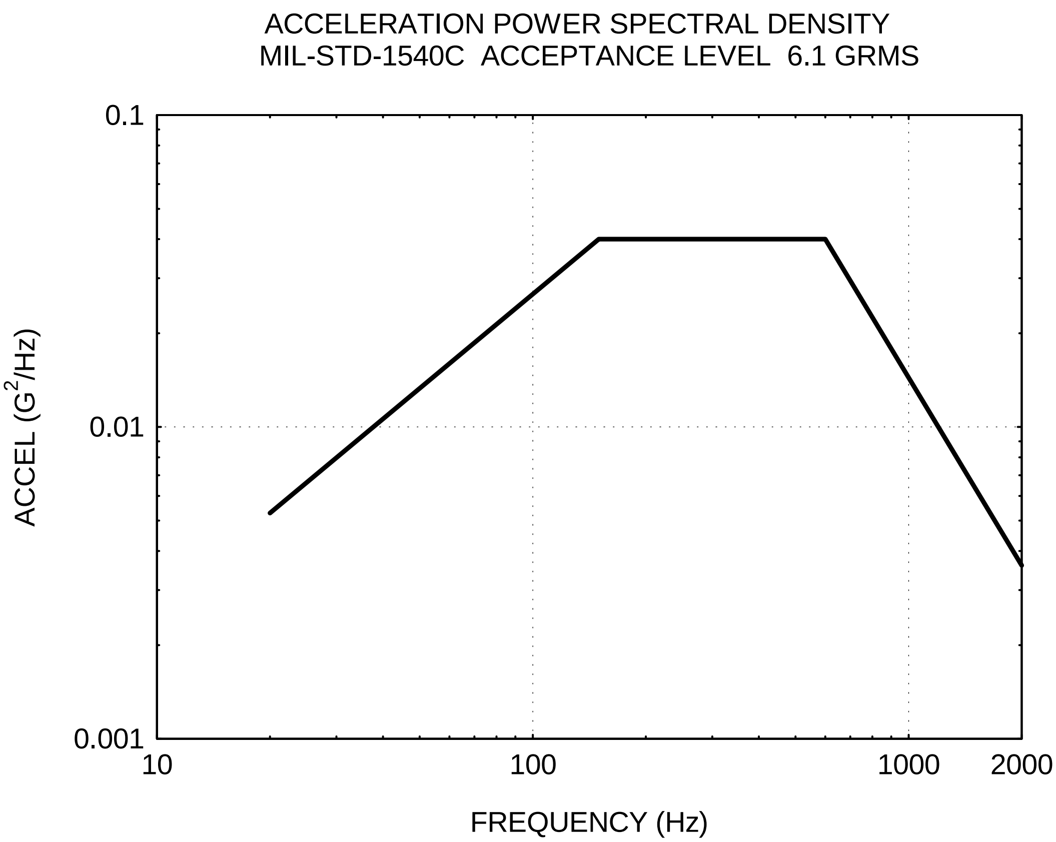


Figure 1.

Table 2. MIL-STD-1540C Acceptance Level, 6.1 GRMS Overall	
Frequency (Hz)	Accel (G^2/Hz)
20	0.0053
150	0.04
600	0.04
2000	0.0036

Now consider an avionics component which is to be tested to the MIL-STD-1540C level. Assume that the component consist of a metal housing and a circuit board. Further assume that the circuit board can be modeled as a single-degree-of-freedom system with an amplification factor of Q=10.

How will the circuit board respond to the input level? The response amplitude depends on the natural frequency. The response power spectral density is shown for three natural frequency cases in Figure 2.

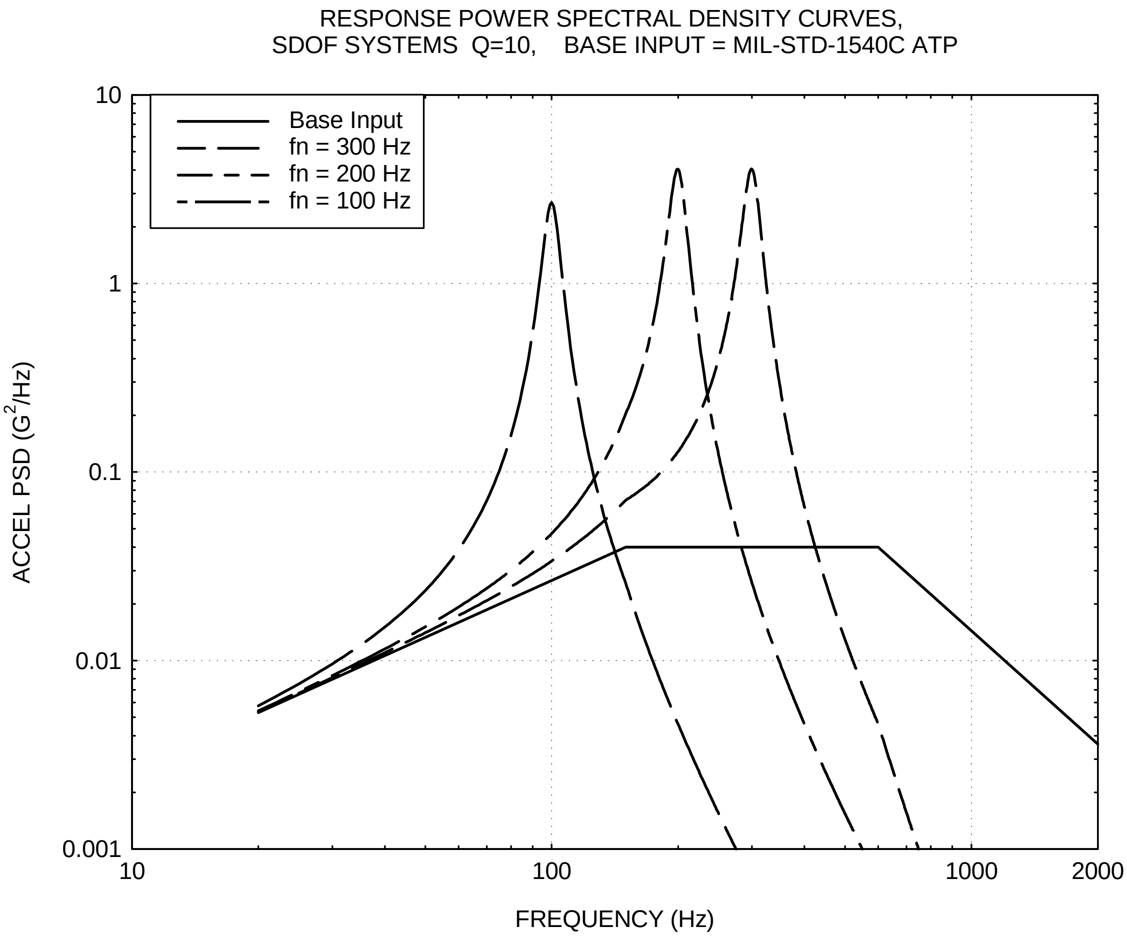


Figure 2.

Each response curve in Figure 2 is calculated using the transfer magnitude function which is embedded in equation (8). The next task is to calculate the area under each response curve. The square root of the area is the GRMS value for the particular natural frequency.

Note that equation (8) performs the complete calculation from the base input level to the GRMS response for each natural frequency. Plotting the individual response curves is optional.

The GRMS values for the response curves in Figure 2 are shown in Table 3.

Table 3. Vibration Response Spectra, Q=10

Natural Frequency (Hz)	Response Accel (GRMS)
100	6.4
200	11.1
300	13.7

These steps are repeated via equation (8) for a family of natural frequencies in order to generate the complete vibration response spectrum, as shown in Figure 3. Note that the curve in Figure 3 passes through the coordinates in Table 3.

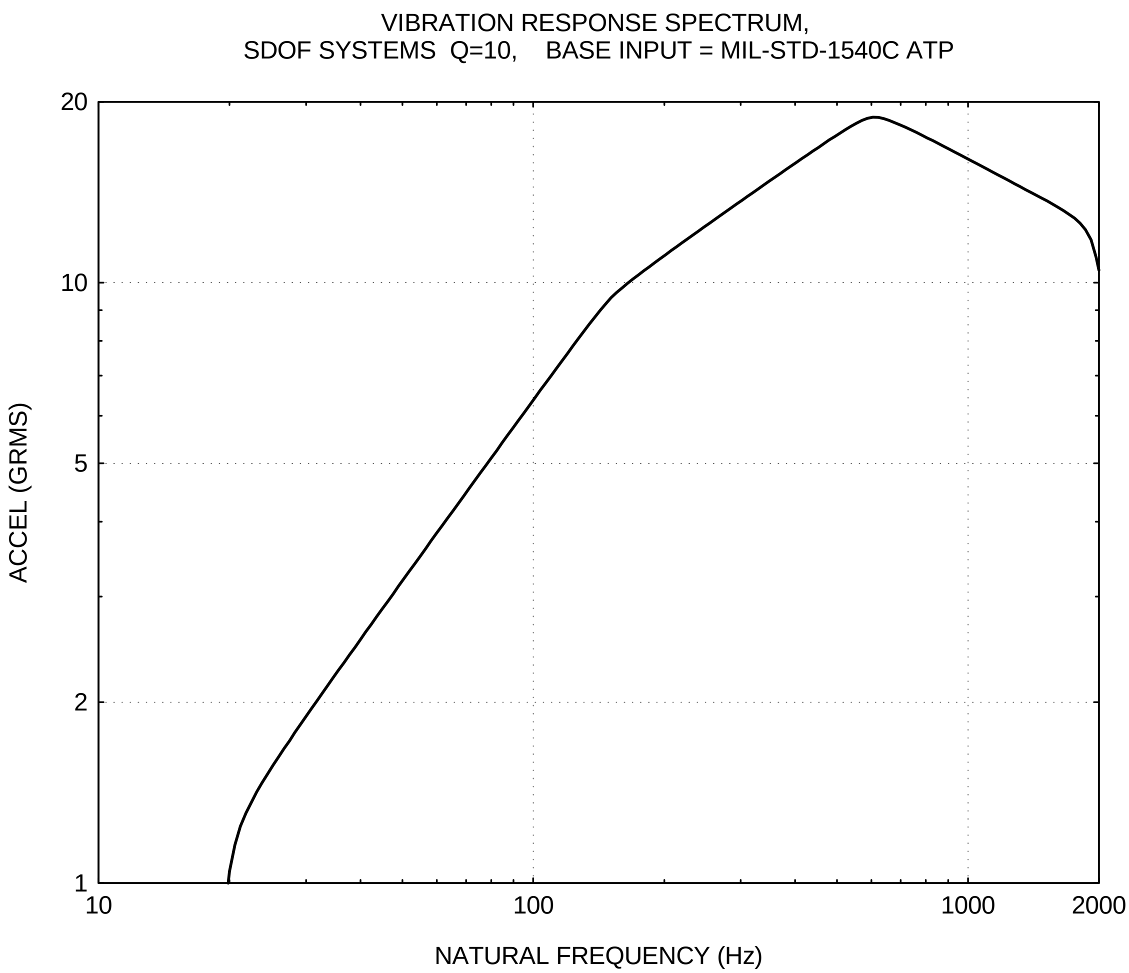


Figure 3.

The curve in Figure 3 can be used for design purposes. It can also be used to evaluate the severity of the test. The worst-case point is (600 Hz, 18.9 GRMS). A design goal might be to avoid this natural frequency.

TEST LEVEL DERIVATION EXAMPLE

Avionics Component

Need for Test Level

The purpose of this example is to show how the vibration response spectrum can be used to derive a component vibration test level from flight data.

Consider an avionics transponder which is to be mounted inside the fuselage of a supersonic fighter aircraft. The component will be subjected to flight vibration from a number of sources. A maximum expected flight level is required so that a test level can be derived. Note that the test level must envelop the maximum expected flight level plus some margin.

Excitation Sources

The aircraft will encounter aerodynamic buffeting as it accelerates through the transonic velocity. Note that shock waves begin to form on the aircraft as it approaches the transonic velocity. These shock waves could excite fuselage skin bending modes. Turbulent boundary layers could likewise excite structural modes.

In addition, the transponder will be excited by structural-borne vibration from the turbofan engines.

Measured Data from Flight Test

A prototype test aircraft is instrumented with accelerometers. It is then flown through a variety of harsh maneuvers. One of the accelerometers is mounted adjacent to the transponder. A power spectral density of the data from this accelerometer is given in Figure 4. This power spectral density is considered as the base input to the component.

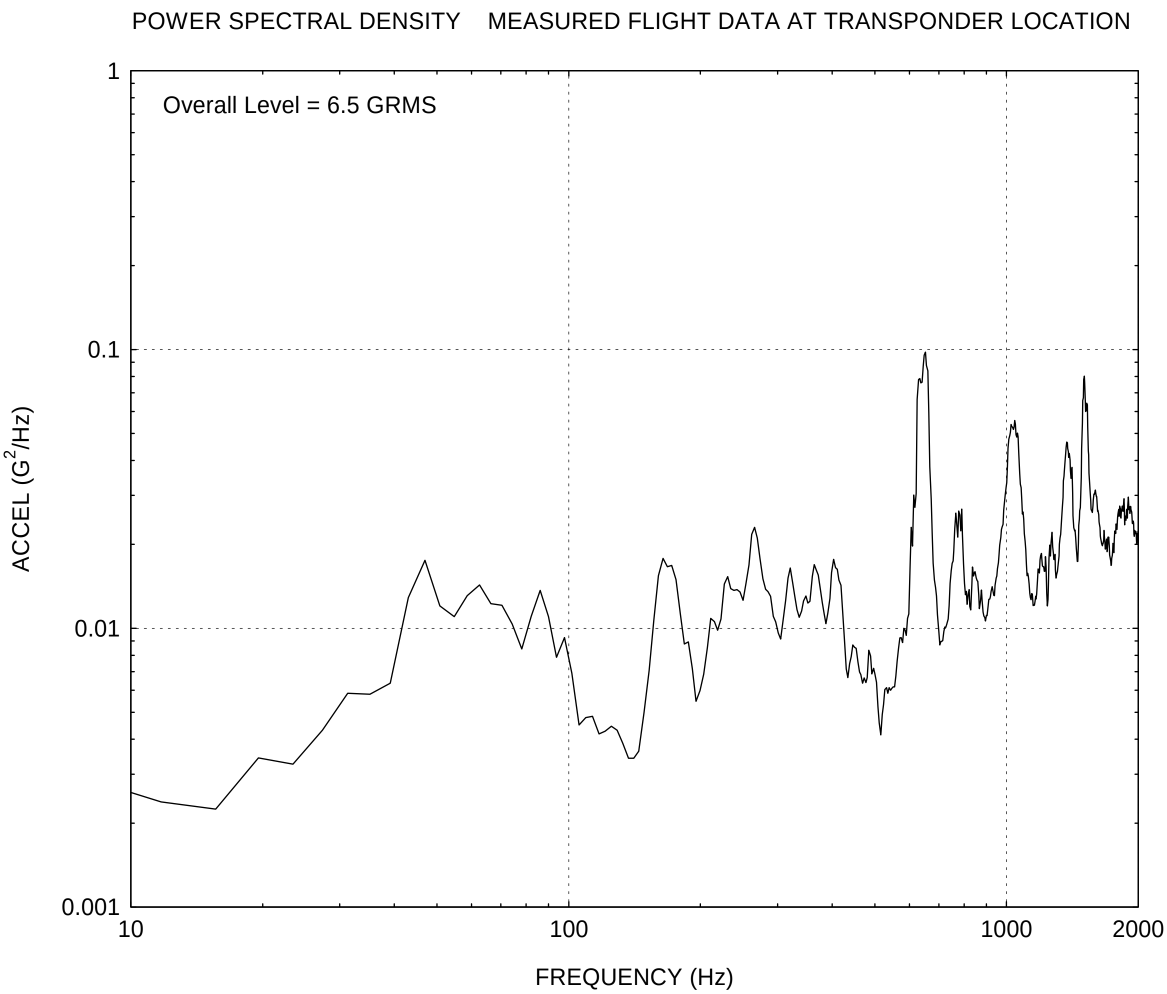


Figure 4.

Proposed Maximum Expected Flight Data

The next step is to derive a maximum expected flight level. The maximum expected flight level does not need to be so finely tailored that it envelops each spectral peak.

Note that there could be some frequency variation in these peaks, for example, as external armaments are reconfigured. Variation could also occur depending on the trajectory of a particular flight. Furthermore, the aircraft might be produced in different version. The naval version could have a stiffened fuselage to endure harsh carrier landings. The stiffened fuselage could have higher natural frequencies than the air force version.

Thus, a more prudent approach is to derive a generic maximum expected flight level consisting of several segments in a power spectral density function.

A proposed maximum expected flight level is given in Figure 5, along with the measured flight data.

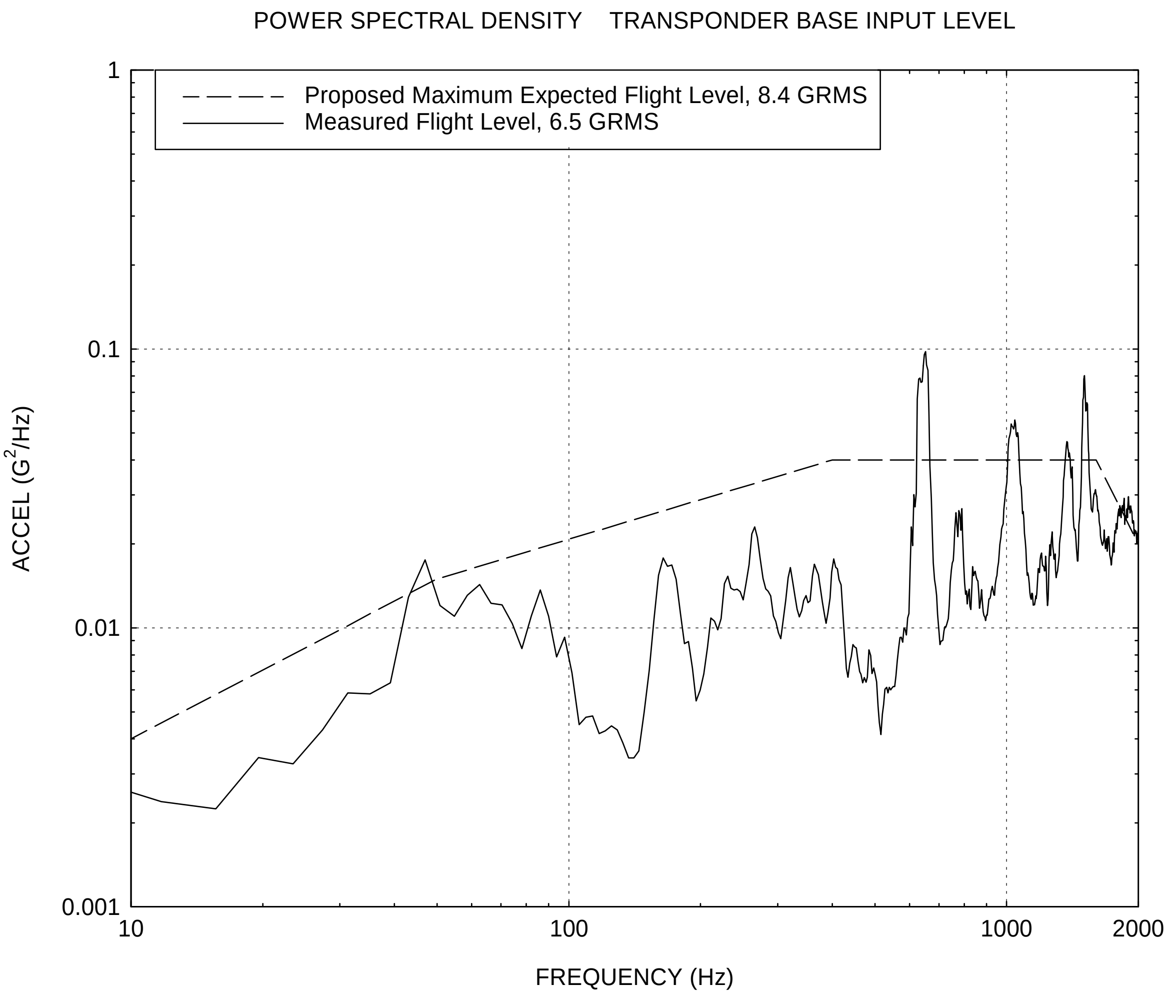


Figure 5.

The proposed flight level has an overall level of 8.4 GRMS, which is higher than the measured flight level of 6.5 GRMS. The overall GRMS level is the square root of the area under the power spectral density curve. It is also the standard deviation of the corresponding time history, assuming a zero mean.

Clearly, the proposed maximum flight level does not envelop every spectral peak of the measured data. In particular, the measured data has a peak at 650 Hz which protrudes above the proposed level. Again, this frequency could shift in a different vehicle configuration.

On the other hand, the vibration response spectrum of the proposed level envelops that of the measured data as shown in Figure 6. The spectra were calculated using equation (8).

The proposed level was in fact designed using equation (8) in a trial-and-error manner. The goal was to design an input power spectral density which would envelop the flight curve in terms of vibration response spectra.

Note that the vibration response spectrum smoothes the flight data peaks. Also note that the X-axis is given in terms of natural frequency.

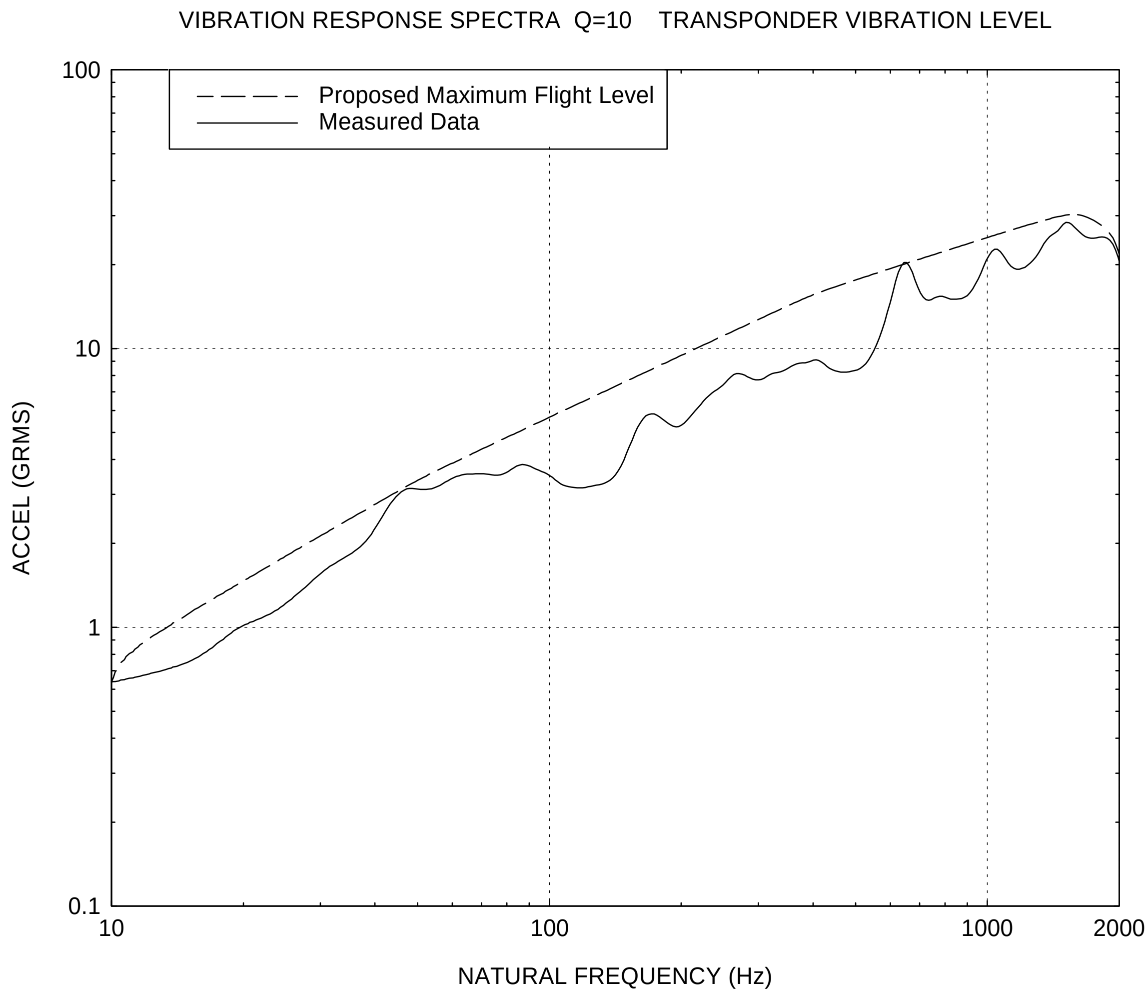


Figure 6.

Test Level Derivation

The proposed level is thus accepted as the maximum expected flight level on the basis on the vibration response spectrum comparison. The breakpoints for the maximum expected level are given in Table 4.

Table 4.
Power Spectral Density,
Maximum Expected Flight Level,
8.4 GRMS Overall

Frequency (Hz)	Accel (G^2/Hz)
10	0.004
50	0.015
400	0.04
1600	0.04
2000	0.02

A test level can now be derived from the maximum expected level by adding appropriate margins and safety factors.

For example, the level could be increased to compensate for the fact that the transponder will only be tested for several minutes on a vibration shaker, even though the unit will be expected to function during several thousand hours of flight time.

The amount of margin also depends on the policies of the vendor, the customer, and government agencies. Note the U.S. Department of Defense has set up certain guidelines and requirements for avionics test levels. The requirements are given in documents such as MIL-STD-1540C and MIL-STD-810E.

Furthermore, two test levels need to be derived. The first level is an acceptance test level. This level must envelop the maximum expected flight level. It must also be sufficient to uncover latent defects in parts and workmanship, such as bad solder joints. Each component must pass an acceptance test before it is mounted in an aircraft.

The other level is the qualification level, which is typically 6 dB higher than the acceptance level. The purpose of the qualification test is to verify the design integrity. Only a few units will be subjected to qualification testing. These units will not be flown.

A derivation of the final test levels is beyond the scope of this report. The purpose of this example was simply to demonstrate the use of the vibration response spectrum in transforming measured flight data into a maximum expected flight level. The maximum expected flight level would then serve as a basis for test levels.

Again, the vibration response spectrum has many other uses, some of which will be included in future revisions of this tutorial.

REFERENCES:

1. T. Irvine, An Introduction to the Shock Response Spectrum Rev O, Vibrationdata Publications, 2000.
2. L. Meirovitch, Analytical Methods in Vibrations, Macmillan, New York, 1967.
3. W. Thomson, Theory of Vibration with Applications 2nd Edition, Prentice Hall, New Jersey, 1981.

APPENDIX A

Acceleration Response Derivation

Recall equation (6) from the main text.

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z = -\ddot{y} \tag{A-1}$$

Now take the Fourier transform of each side

$$\int_{-\infty}^{\infty}\left\{\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z\right\}e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}\{-\ddot{y}\}e^{-j\omega t}\,dt \tag{A-2}$$

Note that the approach used here is rigorous. Simpler approaches are often used in other references.

Let

$$Y(\omega) = \int_{-\infty}^{\infty}\{y(t)\}e^{-j\omega t}\,dt$$

$$X(\omega) = \int_{-\infty}^{\infty}\{x(t)\}e^{-j\omega t}\,dt$$

Now take the Fourier transform of the velocity term

$$\int_{-\infty}^{\infty}\{\dot{z}(t)\}e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}\left\{\frac{dz(t)}{dt}\right\}e^{-j\omega t}\,dt \tag{A-3}$$

Integrate by parts

$$\int_{-\infty}^{\infty}\{\dot{z}(t)\}e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}d\left\{z(t)e^{-j\omega t}\right\} - \int_{-\infty}^{\infty}[z(t)](-j\omega)e^{-j\omega t}dt \tag{A-4}$$

$$\int_{-\infty}^{\infty}\{\dot{z}(t)\}e^{-j\omega t}\,dt = z(t)e^{-j\omega t}\Big|_{-\infty}^{\infty} + (j\omega)\int_{-\infty}^{\infty}z(t)e^{-j\omega t}dt \tag{A-5}$$

$$z(t)e^{-j\omega t}\Big|_{-\infty}^{\infty} = 0 \text{ as t approaches the} \pm\infty \text{ limits.} \tag{A-6}$$

$$\int_{-\infty}^{\infty}\{\dot{z}(t)\}e^{-j\omega t}dt = (j\omega)\int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt \tag{A-7}$$

$$\int_{-\infty}^{\infty}\{\dot{z}(t)\}e^{-j\omega t}dt = (j\omega)X(\omega) \tag{A-8}$$

Furthermore

$$\int_{-\infty}^{\infty}\{\ddot{z}(t)\}e^{-j\omega t}dt = \int_{-\infty}^{\infty}\left\{\frac{d^2z(t)}{dt^2}\right\}e^{-j\omega t}dt \tag{A-9}$$

$$\int_{-\infty}^{\infty}\{\ddot{z}(t)\}e^{-j\omega t}dt = \int_{-\infty}^{\infty} d\left\{\frac{dz(t)}{dt}e^{-j\omega t}\right\} - \int_{-\infty}^{\infty}[\frac{dz(t)}{dt}(-j\omega)e^{-j\omega t}dt \tag{A-10}$$

$$\int_{-\infty}^{\infty}\{\ddot{z}(t)\}e^{-j\omega t}dt = \frac{dz(t)}{dt}e^{-j\omega t}\Big|_{-\infty}^{\infty} + (j\omega)\int_{-\infty}^{\infty}\frac{dz(t)}{dt}e^{-j\omega t}dt \tag{A-11}$$

$$\frac{dz(t)}{dt}e^{-j\omega t}\Big|_{-\infty}^{\infty} = 0 \text{ as t approaches the } \pm\infty \text{ limits.} \tag{A-12}$$

$$\int_{-\infty}^{\infty}\{\ddot{z}(t)\}e^{-j\omega t}dt = (j\omega)\int_{-\infty}^{\infty}\frac{dz(t)}{dt}e^{-j\omega t}dt \tag{A-13}$$

$$\int_{-\infty}^{\infty}\{\ddot{z}(t)\}e^{-j\omega t}dt = (j\omega)(j\omega)Z(\omega) \tag{A-14}$$

$$\int_{-\infty}^{\infty}\{\ddot{z}(t)\}e^{-j\omega t}dt = -\omega^2 Z(\omega) \tag{A-15}$$

Recall

$$\int_{-\infty}^{\infty}\left\{\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z\right\}e^{-j\omega t}dt = \int_{-\infty}^{\infty}\{-\ddot{y}\}e^{-j\omega t}dt \tag{A-16}$$

Let the subscript A denote acceleration. By substitution,

$$-\omega^2 Z(\omega) + j\omega(2\xi\omega_n)Z(\omega) + \omega_n^2 Z(\omega) = -Y_A(\omega) \tag{A-17}$$

$$\left\lfloor (\omega_n^2 - \omega^2) + j2\xi\omega\omega_n \right\rfloor Z(\omega) = -Y_A(\omega) \tag{A-18}$$

$$Z(\omega) = \frac{-Y_A(\omega)}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n} \tag{A-19}$$

$$Z_A(\omega) = -\omega^2 Z(\omega) \tag{A-20}$$

$$Z_A(\omega) = \frac{\omega^2 Y_A(\omega)}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n} \tag{A-21}$$

The relative acceleration equation can be expressed in terms of Fourier transforms as

$$Z_A(\omega) = X_A(\omega) - Y_A(\omega) \tag{A-22}$$

$$X_A(\omega) = Z_A(\omega) + Y_A(\omega) \tag{A-23}$$

$$X_A(\omega) = \frac{\omega^2 Y_A(\omega)}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n} + Y_A(\omega) \tag{A-24}$$

$$X_A(\omega) = \frac{[\omega^2 + (\omega_n^2 - \omega^2) + j2\xi\omega\omega_n]Y_A(\omega)}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n} \tag{A-25}$$

$$X_A(\omega) = \frac{[\omega_n^2 + j2\xi\omega\omega_n]Y_A(\omega)}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n} \tag{A-26}$$

Multiply each side by its complex conjugate

$$X_A(\omega)X_A^*(\omega) = \frac{[\omega_n^2 + j2\xi\omega\omega_n][\omega_n^2 - j2\xi\omega\omega_n]Y_A(\omega)Y_A^*(\omega)}{\left[(\omega_n^2-\omega^2)+j2\xi\omega\omega_n\right]\left[(\omega_n^2-\omega^2)-j2\xi\omega\omega_n\right]} \tag{A-27}$$

$$X_A(\omega)X_A^*(\omega) = \frac{[\omega_n^4 + (2\xi\omega\omega_n)^2]Y_A(\omega)Y_A^*(\omega)}{(\omega_n^2-\omega^2)^2+(2\xi\omega\omega_n)^2} \tag{A-28}$$

$$X_A(\omega)X_A^*(\omega) = \frac{\omega_n^2[\omega_n^2 + (2\xi\omega)^2]Y_A(\omega)Y_A^*(\omega)}{(\omega_n^2-\omega^2)^2+(2\xi\omega\omega_n)^2} \tag{A-29}$$

The Fourier transforms are converted into power spectral densities using the method shown in Reference 3, where T is the duration.

$$\lim_{T\to\infty} X_A(\omega)X_A^*(\omega)/T = X_{APSD}(\omega) \tag{A-30}$$

$$\lim_{T\to\infty} Y_A(\omega)Y_A^*(\omega)/T = Y_{APSD}(\omega) \tag{A-31}$$

$$X_{APSD}(\omega) = \frac{\omega_n^2[\omega_n^2 + (2\xi\omega)^2]Y_{APSD}(\omega)}{(\omega_n^2-\omega^2)^2+(2\xi\omega\omega_n)^2} \tag{A-32}$$

Equation (A-32) can be transformed as a function of frequency f as follows

$$\hat{X}_{APSD}(f) = \frac{f_n^2[f_n^2 + (2\xi f)^2]\hat{Y}_{APSD}(f)}{(f_n^2-f^2)^2+(2\xi f f_n)^2} \tag{A-33}$$

Divide each side by f_n^4

$$\hat{X}_{APSD}(f) = \frac{[1 + (2\xi f/f_n)^2]\hat{Y}_{APSD}(f)}{(1-(f/f_n)^2)^2+(2\xi f/f_n)^2} \tag{A-34}$$

Let $\rho = f / f_n$

$$\hat{X}_{APSD}(f) = \frac{1+(2\xi\rho)^2}{(1-\rho^2)^2+(2\xi\rho)^2}\,\hat{Y}_{APSD}(f), \quad \rho = f/f_n \tag{A-35}$$

The overall response $\ddot{x}_{GRMS}$ can be obtained by integrating $\hat{X}_{APSD}(f)$ across the frequency spectrum and then taking the square root of the area.

$$\ddot{x}_{GRMS}(f_n,\xi) = \sqrt{\int_0^{\infty}\left[\frac{1+(2\xi\rho)^2}{(1-\rho^2)^2+(2\xi\rho)^2}\right]\hat{Y}_{APSD}(f)df}\ , \quad \rho = f/f_n \tag{A-36}$$

Often the function $\hat{Y}_{APSD}(f)$ is available only in a digital format. The integral can thus be replaced by a summation. Note that Δf_i is usually a constant.

$$\ddot{x}_{GRMS}(f_n,\xi) = \sqrt{\sum_{i=1}^{N}\left[\frac{1+(2\xi\rho_i)^2}{(1-\rho_i{}^2)^2+(2\xi\rho_i)^2}\right]\hat{Y}_{APSD}(f_i)\Delta f_i}\ , \quad \rho_i = f_i/f_n \tag{A-37}$$

APPENDIX B

<u>Relative Displacement Derivation</u>

Recall

$$Z(\omega) = \frac{-Y_A(\omega)}{(\omega_n{}^2 - \omega^2) + j2\xi\omega\omega_n} \tag{B-1}$$

Multiply by the complex conjugate

$$Z(\omega)\,Z^*(\omega) = \left[\frac{1}{(\omega_n{}^2 - \omega^2) + j2\xi\omega\omega_n}\right]\left[\frac{1}{(\omega_n{}^2 - \omega^2) - j2\xi\omega\omega_n}\right] Y_A(\omega) Y_A{}^*(\omega) \tag{B-2}$$

$$Z(\omega)\,Z^*(\omega) = \left[\frac{1}{(\omega_n{}^2 - \omega^2)^2 + (2\xi\omega\omega_n)^2}\right] Y_A(\omega) Y_A{}^*(\omega) \tag{B-3}$$

$$Z_{PSD}(\omega) = \left[\frac{1}{(\omega_n{}^2 - \omega^2)^2 + (2\xi\omega\omega_n)^2}\right] Y_{APSD}(\omega) \tag{B-4}$$

$$\hat{Z}_{PSD}(f) = \frac{1}{[2\pi]^4}\left[\frac{1}{(f_n{}^2 - f^2)^2 + (2\xi f f_n)^2}\right] \hat{Y}_{APSD}(f) \tag{B-5}$$

The overall relative displacement RMS is

$$Z_{RMS}(f_n,\xi) = \sqrt{\int_0^{\infty} \frac{1}{[2\pi]^4}\left[\frac{1}{f_n{}^2 - f^2)^2 + (2\xi f f_n)^2}\right] \hat{Y}_{APSD}(f)\, df}$$

(B-6)

$$Z_{RMS}(f_n,\xi) = \frac{1}{[2\pi]^2}\sqrt{\int_0^{\infty} \left[\frac{1}{(f_n{}^2 - f^2)^2 + (2\xi f f_n)^2}\right] \hat{Y}_{APSD}(f)\, df}$$

(B-7)

The relative displacement RMS value for digital data is

$$Z_{RMS}(f_n,\xi) = \frac{1}{[2\pi]^2}\sqrt{\sum_{i=1}^{N} \left[\frac{1}{(f_n{}^2 - f_i{}^2)^2 + (2\xi f_i f_n)^2}\right] \hat{Y}_{APSD}(f_i)\Delta f_i}$$

(B-8)